Calculation of x-ray spectra emerging from an x-ray tube. Part I.
Electron penetration characteristics in x-ray targets

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The penetration characteristics of electron beams into x-ray targets are investigated for incident electron kinetic energies in the range 50–150 keV. The frequency densities of electrons penetrating to a depth x in a target, with a fraction of initial kinetic energy, u, are calculated using Monte Carlo methods for beam energies of 50, 80, 100, 120 and 150 keV in a tungsten target. The frequency densities for 100 keV electrons in Al, Mo and Re targets are also calculated. A mixture of simple modeling with equations and interpolation from data is used to generalize the calculations in tungsten. Where possible, parameters derived from the Monte Carlo data are compared to experimental measurements. Previous electron transport approximations in the semiempirical models of other authors are discussed and related to this work. In particular, the crudity of the use of the Thomson-Whiddington law to describe electron penetration and energy loss is highlighted. The results presented here may be used towards calculating the target self-attenuation correction for bremsstrahlung photons emitted within a tungsten target. © 2007 American Association of Physicists in Medicine. [DOI: 10.1118/1.2734725]

Key words: spectral modeling, kilovoltage, x-ray production, Monte Carlo, radiation transport

I. INTRODUCTION

The accurate prediction of the photon spectrum emerging from an x-ray tube is important in imaging and therapy contexts, as the spectrum affects imaging properties and patient dose. An understanding of the penetration characteristics of beams of electrons into thick x-ray targets is necessary, as the depth of the associated x-ray production affects this spectrum via target self-filtration.

Determinations of the properties of electron transmission, using metal films, were made by Whiddington in 1912. That author found, empirically, that the square of the most probable energy of an electron, emerging from a film of thickness x, is approximately linearly dependent on x. The energy spectra of electron beams transmitted through slabs of media were investigated by later workers. A formula of the same form as for the most probable energy was shown to describe the mean energy, \( \langle T \rangle \), very well. This is referred to in the literature as the Thomson-Whiddington law and may be written

\[
\langle T(x) \rangle^2 = T_0^2 - Cpx,
\]

where \( T_0 \) is the kinetic energy of the incident electrons, p is target density and C is known as a Thomson-Whiddington constant. This last “constant” is approximately material independent, but increases slowly with \( T_0 \). Semiempirical models of x-ray production have been proposed that make use of such a Thomson-Whiddington law to relate an electron’s energy to its penetration depth as a simple one-to-one mapping. Some of these models, among other approaches, have been applied in a healthcare setting. The values for the Thomson-Whiddington constants used in these latter cases were those calculated by Birch and Marshall, using the extrapolated transmission range data of Katz and Penfold. The distribution in the energies of electrons at depth was ignored and backscatter was neglected.

This work addresses these issues using Monte Carlo electron transport. While the penetration characteristics of electrons have been investigated previously using Monte Carlo methods, e.g., Bishop or Sundararaman, Prasad and Vora, more recent articles have treated the entire process of x-ray generation, including the bremsstrahlung emission, using Monte Carlo, e.g., Verhaegen et al. or Ay et al. With modern computing power and the careful use of variance reduction techniques, the full Monte Carlo simulation of an x-ray unit may, perhaps, be performed in a matter of minutes or less. However, the separation of the process of bremsstrahlung generation into two steps, electron penetration into a target and bremsstrahlung emission, can provide insight into the nature of beam transport in an x-ray target and the affect that this has on an emerging spectra. Examinations of the sufficiency of theoretical results for the differential bremsstrahlung cross section are also readily made. This article, Part I, is the first of two papers. Here, the transport of electron beams corresponding to tube potentials of 50–150 kVp is simulated in an x-ray target using Monte Carlo methods. A second paper, Part II, presents spectral predictions using these results and bremsstrahlung cross sections.

The primary objective of this paper, Part I, is to introduce, calculate and parameterize the “joint frequency density” describing electron penetration. This quantity, \( f(u,x) \), is the number density of electrons that reach a depth x, with a fraction of the incident kinetic energy, u. This joint frequency density may be decomposed into a planar survival
frequency, \( \eta_{pf}(x) \), which is the frequency with which electrons reach a depth \( x \), and the probability, \( P(u|x) \), that an electron at that depth possesses an fraction of its initial energy, \( u \). Then,

\[
f(u,x) = \eta_{pf}(x)P(u|x).
\]  

(2)

The probability \( P(u|x) \) will be referred to as the conditional probability function (CPF). Further, since an x-ray emission occurs within a thick target, it is convenient to further decompose Eq. (2) into a first-pass (\( F \)) and a multiple-pass (\( M \)) component such that

\[
f(u,x) = \eta_{pf}(x)P_F(u|x) + \eta_{M}(x)P_M(u|x).
\]  

(3)

The first-pass component corresponds to the quantity measured in transmission measurements, such as Whiddington’s, through a slab of thickness \( x \). The multiple-pass component is that one that is present due to backscatter from the material below the depth \( x \). For the simple treatment developed here, it was necessary to demonstrate that a state of diffusion was reached rapidly in a target so that the angle of an electron at depth compared to its incident direction is independent of and uncorrelated to its kinetic energy.

II. THEORY AND METHOD

A. Scattering regions

Energetic electrons penetrating into thick high-Z materials undergo many scatters before coming to rest and several interaction processes are relevant.\(^{20}\) The paths of the electrons are tortuous due to multiple deflections through elastic scattering. Consider electrons with the same initial kinetic energy incident normal to a target surface. Relatively frequently an electron is kicked backwards in an interaction with the nucleus of an atom, a process which is sufficiently described by Rutherford or Mott scattering, but the majority of the scattering events are small angle deflections, in which the atomic cloud appreciably screens the nuclear potential. The first extremely thin layer of the target, corresponding to a few elastic mean free paths of penetration, is called the plural scattering region, and is normally assumed to apply where the number of scattering events is less than 20. This corresponds to \( \sim 0.1 \mu m \) in tungsten for electrons with kinetic energies of tens of keV.\(^{21}\) By the time a few tens of mean free paths have been traversed, statistical averaging over an ensemble of electrons results in a Gaussian angular distribution (for scattering angles less than \( \sim 20^\circ \)), with the most probable angle of an electron increasing with penetration depth. This is referred to as the multiple-scattering region. The final scattering region, diffusion, has been defined as the state in which the angular distribution of an electron beam penetrating into a material no longer changes with depth.\(^{20}\) In the elastic scattering of electrons from nuclei, energy loss is negligible, since the mass of a nucleus can be assumed infinite. In between elastic scattering events, however, inelastic scattering from electrons bound within the target atoms occurs, resulting in energy loss, ionization, and knock-on electrons. The relative frequency of inelastic to elastic scattering events is dependent on target atomic number and the electron energy, however, the elastic cross section exceeds the inelastic for high-Z targets in the energy range of interest.

B. Geometry and angular distributions

The path of an electron in a target is depicted in Fig. 1. In this instance the primary electron crosses a plane at a depth, \( x \), three times before coming to rest. The path length traveled at the \( n \)th crossing is denoted by \( \ell_n \). The first crossing of the electron through the plane would be assigned, in an appropriate energy bin, as a “first-pass” and the second and third, in their appropriate energy bins, as “multiple-pass” contributions. The electron velocity after a path-length \( l \), is \( V(l) \). The incident electron velocity vector, \( V_0 \), is aligned with the positive \( X \) axis, normal to the surface of a semi-infinite target. The angle between these two vectors is the scattering angle, \( \theta(l) \). Account must also be taken of knock-on electrons; \( \delta \)-rays generated at a depth less than \( x \), passing through \( x \), may contribute to the first-pass component and \( \delta \) rays generated at a depth beyond \( x \) that are scattered backwards through the plane at that depth are necessarily contributions to the multiple-pass component.

Within the target there will be a straggling in an electron’s lateral displacement in the \( Y-Z \) plane, relative to its initial position, due to scatter. For a beam of a large enough area and sufficient intensity this may be neglected from the point of view of the origin of x-ray emissions. For every electron displaced from a \( Y-Z \) coordinate, there is one of the same kinetic energy to replace it at that location, excepting where an electron is close enough to the beam edge that its range may take it outside the original beam area. An electron beam used in general diagnostic x-ray tube has the approximate dimensions: \( L \times L = 0.1 \times 0.1 \text{ cm}^2 \) (see Ref. 22). The typical penetration range for an electron in tungsten is \( r \sim 10^{-3} \text{ cm} \). The fraction of electrons within range of edge effects is therefore small, being: \( 4 \times L \times r/L^2 = 0.04 \). It is therefore reasonable to treat electrons in a beam as if they had no lateral straggling from a macroscopic perspective, i.e., from that of an ensemble of electrons emitting x rays. Further, the beam spot size increases little with depth and can be considered approximately the same as at the surface.

While an electron’s lateral straggling may be ignored macroscopically, it is crucial to realize that its path length, \( L \), will be greater than the sum of its paths in the \( X \) direction. That is, the average number of bremsstrahlung photons emit-
ted per electron per unit distance penetrated will be dependent on the angular distribution of the electron beam. This may be parameterized in a path-length correction. Azimuthal symmetry will be assumed in the angular distribution and the appropriate spherical distribution must be projected onto a $Y$-$Z$ plane. It is therefore convenient to have an angular distribution at diffusion, $h(\theta)$, with an associated normalization,

$$\int_0^\pi d\theta h(\theta) = 1. \quad (4)$$

However, the sign of an electron’s velocity vector with respect to the $X$ axis is immaterial to interaction probability and for certain purposes it is convenient to add these contributions together to form a new distribution

$$g(\theta) = h(\theta) + h(\pi - \theta), \quad (5)$$

where, now, $0 \leq \theta < \pi/2$. The diffusion angular distribution, in a scoring plane, for pure elastic scattering in an infinite medium is

$$g_d(\theta) = 2 \cos \theta \sin \theta. \quad (6)$$

The factor $\sin \theta$ projects a spherical fluence distribution onto an annulus on the surface of a sphere and the factor $\cos \theta$ further projects onto the $Y$-$Z$ plane.

Although an individual electron scatters many times between two planes separated by a distance of a fraction of a mm, if the beam is in diffusion, for each electron scattered at a new angle, there is, on average, another electron of the same energy that scatters to replace it. The path-length correction then factorizes out into a multiplicitive diversion factor. The mean diversion, $d_v$, will be defined

$$d_v = \left( \frac{\Delta l}{\Delta x} \right)_{\text{diffusion}} \rightarrow \left( \frac{1}{\cos \theta} \right)_{g(\theta)} = \int_0^{\pi/2} d\theta \frac{g(\theta,x)}{\cos \theta}. \quad (7)$$

This factor, assuming Eq. (6) is valid and that diffusion is reached instantaneously, takes the value, $d_v=2$.

### C. Monte Carlo simulation

BEAMnrc was used to simulate the penetration of normally incident monoenergetic electrons into a pure semi-infinite tungsten target using the EGSnrc Monte Carlo code. Each simulation consisted of $4 \times 10^6$ electron histories and required approximately 1 hour of CPU time. The exact boundary crossing algorithm was used for electron transport and single scattering was imposed by the choice of an appropriately large skin depth ($10^{10}$ mean free paths). The low incident electron kinetic energies of 50–150 keV made electron transport practicable without the condensation of multiple elastic scattering events into single steps. Spin effects were turned on. The values of $AE$ and $AP$, determining the smallest energy-loss increment for stochastic treatment of inelastic collisions and bremsstrahlung, respectively, were set to 512 and 1 keV. The values of $UE$ and $UP$, determining the high-energy cutoffs, for electrons and photons, respectively, were set to 1012 and 500 keV. In all simulations $\text{ECUT}$ was set to 521 keV, such that each electron was transported until its kinetic energy fell below 10 keV. As very few bremsstrahlung photons of energy less than 10 keV escape an x-ray unit, it was considered unnecessary to pursue electrons’ progress beyond this cutoff. Phase-space files were generated at 0.5 or 1.0 $\mu$m increments of depths, with the maximum depth scored depending on the incident energy and varying between 2.5 and 14 $\mu$m. Five electron kinetic energies were investigated: 50, 80, 100, 120 and 150 keV. Analysis of the phase-space files was performed using the BEAMnrc analysis code. The first-pass contributions to a phase-space file were analyzed separately to the multiple-pass frequency, using “latching.” Simulations for aluminium, molybdenum, and rhenium were also obtained, for $T_0=100$ keV only. These materials provide examples of low, medium and high-Z elements, respectively, to illustrate the effects of atomic number on the results. Molybdenum and rhenium are both commonly used materials in x-ray targets and many electron beam transmission measurements have used aluminium as their attenuating material.

### D. First-pass frequency

Under diffusion conditions, the rate of decay of the first-pass frequency with penetration depth is expected to be proportional to the first-pass frequency itself:

$$d \eta_F = - \sum \eta_F dx,$$  \quad (8)

where $\Sigma$ is an attenuation coefficient related to the elastic and inelastic cross sections. A plausible form for $\eta_F$ can be found heuristically without explicit knowledge of $\Sigma$. In the nonrelativistic limit, both the Mott cross section describing elastic scatter and the Møller cross section describing inelastic scatter have an inverse square dependence on the kinetic energy of an electron. Hypothesizing an inverse square dependence for $\Sigma$:

$$\eta_F(x) \sim \exp\left( - \Sigma_0 \int_0^x \frac{T_0^2}{(T(x'))^2} dx' \right)$$

$$= \exp\left( - \Sigma_0 \int_0^x \frac{dx'}{(\langle u(x') \rangle_F)^2} \right), \quad (9)$$

where

$$\langle u(x) \rangle_F = \langle T(x) \rangle_F T_0 = \int_0^1 \langle u P_F(x) \rangle_F du,$$  \quad (10)

and where $\Lambda=10$ keV/$T_0$ and $\Sigma_0$ is an attenuation coefficient for electron with a kinetic energy, $T_0$. If $\langle u(x) \rangle_F^2$ is replaced with the Thomson-Whiddington prediction, $u_{TW}(x)^2$, using Eq. (1), then after integration,

$$\eta_F = (u_{TW}^2)^\Gamma.$$

The constant $\Gamma=\Sigma_0 R_{TW}/\rho$ is characteristic of the material and $R_{TW}$ [mg cm$^{-2}$] is the Thomson-Whiddington range. This range is that as defined by the Thomson-Whiddington law: $R_{TW} = T_0^2 / C$, where $C$ is the Thomson-Whiddington constant. This range should not be confused with the extrapo-
lated range, $R_{ex}$ [mg cm$^{-2}$], which is defined to be the value of intersection through the $x$ axis of a tangent to the $\eta_F$ curve. These two quantities were equated in the work of Birch and Marshall$^8$ and, hence, in the work dependent upon it.$^8$ Care should be taken in equating these ranges with experiments.$^9$–$10$ Care should be taken in equating these ranges with experiments.$^9$–$10$ Care should be taken in equating these ranges with experiments.$^9$–$10$

E. Multiple-pass frequency

In a thick target, such as that in an x-ray tube, the backscatter from the deeper material must be accounted for at any given depth, $px$. The multiple-pass frequency, $\eta_M$, may be written as the sum of two contributions: the number of electrons moving forwards ($\eta_F$) and the number moving backwards ($\eta_n$). The forward and backward-moving numbers may be treated through the approximation of geometric series.$^{28,29}$ Doing so,

$$\eta = \eta_F B (1 + (BF)^{1}}{1 - FB}$$  \hspace{1cm} (12)

and

$$\eta_n = \eta_n F,$$  \hspace{1cm} (13)

where $B$ and $F$ are the backscatter and forward scatter fractions, respectively. $B$ is the scatter back from the semi-infinite slab below the plane of interest and $F$ is the forward scatter from the slab of finite thickness, $x$, above. The multiple-pass frequency is then

$$\eta_M = \eta_F + \eta_n = \eta_F B \left( \frac{F + 1}{1 - FB} \right)$$  \hspace{1cm} (14)

and the planar survival frequency is, $\eta_0 = \eta_F + \eta_M = \eta_F + \eta_n + \eta_n$. A related quantity is the planar survival current, $\eta_c = \eta_F + \eta_c + \eta_n$. While the survival current is constrained to satisfy $\eta_c \leq 1$, the survival frequency has no such constraint and, for this reason, it is to be interpreted as a frequency and not a survival probability.

At the target surface, necessarily,

$$F(\rho x)|_{\rho x = 0} = 0,$$

and at sufficient depth,

$$F(\rho x) \sim B(\rho x) = B_d,$$

where $B_d$ is the backscatter fraction at diffusion. The simplest parameterization for the forward scatter between zero thickness and asymptotically large depths is, therefore,

$$F(\rho x) = B_{df} [1 - \exp(-K\rho x/R_{TW})],$$  \hspace{1cm} (15)

where $K$ is a dimensionless constant. This functional form exhibits an initial linear rise with $\rho x$ at small values of the exponent. This behavior has been observed in backscatter experiments.$^{28}$

The angular distribution of the beam disperses from the incident $\delta$ function towards the diffusion distribution as it penetrates and, consequently, the backscatter factor initially increases with depth. Choosing the simplest parameterization to describe the transitory variation in the backscatter,

$$B(\rho x) = B_0 + (B_d - B_0)[1 - \exp(-K\rho x/R_{TW})],$$  \hspace{1cm} (16)

where $B_0$ is the backscatter fraction at the surface.

F. Conditional probability functions

Theoretical quantities related to the CPFs, $P_f(u|x)$ and $P_M(u|x)$, exist, such as Landau’s and Vavilov’s straggling functions.$^{30–32}$ These straggling functions are defined in a nontrivial form, however, and furthermore, the functions are defined in terms of path length rather than penetration depth. An empirical approach to describing the CPFs is preferred here. The first- and multiple-pass components are treated separately, since the first-pass distribution is more amenable to experimental measurement, as it corresponds to the energy distribution of a beam exiting a slab of material, and some comparison to experimental findings can be made. The energy spectra for all depths and incident energies, for both the first-pass and multiple components, were generated from the Monte Carlo data in 2 keV bins over the ranges $10 < T < T_0$ keV. The CPFs were obtained by normalizing the integral of these spectra over $u$ to unity. A function was written in MATLAB (MathWorks Inc., Natwick, MA) to linearly interpolate between the data sets in the $u$ and $x$ dimensions. For extrapolation of a CPF to an arbitrary initial kinetic energy, $T_0$, away from the values calculated, the following simple scaling rule is suggested:

$$P_i(u|x; T_0) = P_i(u|xf; T_0),$$  \hspace{1cm} (17)

where $i \in \{F, M\}$ and

$$f = R_{TW}(T_0)/R_{TW}(\tilde{T}_0).$$  \hspace{1cm} (18)

An approximate scaling in energy distributions has been identified elsewhere.$^{27}$

III. RESULTS

A. Validation of assumptions

Monte Carlo angular distributions for a 100 keV beam are presented in Fig. 2(a) at four depths. By a density-scaled depth of 2 mg cm$^{-2}$, equivalent to approximately 1 $\mu$m in tungsten, the distribution has converged. Figure 2(b) shows the planar survival frequency for those 100 keV electrons: it has decreased little at a depth of 2 mg cm$^{-2}$. Diffusion dominates then for the majority of the survival-frequency curve. Therefore, an electron beam of energy $\sim$100 keV, incident upon a high-Z anode, reaches diffusion early in its journey into the target.

Figure 3 presents the Monte Carlo results for the diffusion angular distributions of 50, 80, 100, 120 and 150 keV beams. The curves lie close together. The diffusion angular distribution is, to a good approximation, independent of incident electron energy in the range of interest. The close correspon-
The dependence between the curves permits the use of the 100 keV angular distribution as a “universal distribution” for all the energies.

Figure 4 shows the CPF, as defined by Eq. (14), for 100 keV incident electrons having penetrated to a depth of 3 μm (a density-scaled depth of 5.79 mg cm\(^{-2}\)). The planar CPF was derived from the energy spectrum of planar electron fluence at depth, using BEAMDP analyses of Monte Carlo phase-space files. The path-corrected CPF, also shown, is defined by

\[
P_{\text{Cor}}(u|x) = \frac{1}{d_{DP}} \int \frac{dP}{u} \times \left(\frac{1}{\cos \theta_j(u)}\right) du,
\]

where

\[
d_{DP} = \int \left[ P(u|x) \times \left(\frac{1}{\cos \theta_j(u)}\right)\right] du
\]

and \((1/\cos \theta_j(u))\) is the mean of the reciprocal of the cosine of the scattering angle, for electrons with an energy fraction, \(u\). This second CPF was calculated from the “real fluence” as defined in the BEAMDP Users’ Manual.\(^{26}\) The value of \(d_{DP}\) was found to be 1.88 for the example shown. The similarity in the shape of the two curves suggests that the scattering angle of an electron at depth, is, to an excellent approximation, independent of the energy loss of a particle. This is in agreement with the sparse experimental literature available.\(^{20}\)

### B. Angular distribution

Figure 5 displays the angular distribution \(g(\theta_j)\), generated by Monte Carlo simulation in 2° bins, and the theoretical curve using Eq. (6). The agreement is good and hence \(d_s\) = 2.00 is a reasonable approximation for the mean diversion. This is slightly larger than the value of 1.88 \((d_{DP})\) arrived at by another method in the previous subsection. This discrepancy is partly due to the fact that BEAMDP limits the obliquity with respect to the plane to less than 85° in calculating the real fluence. This cutoff is present to avoid large statistical fluctuations, due to the fact that the number of particles crossing a plane is asymptotically zero as \(\theta_s \to \pi/2\). However, particle diversions approach infinity asymptotically in this limit, accentuating this contribution to the diversion factor such that it may not be completely negligible. To compare diversion estimates fairly, Eq. (7) should be modified to give
Thomson-Whiddington constants /H20849 curves were generated using the fit /H20849 where again Eq. (6) also shown (solid curve).

\[ d_{85\%} = \int_{0^\circ}^{85^\circ} \frac{g(\theta)}{\cos \theta} d\theta + \frac{1}{\cos 85^\circ} \int_{85^\circ}^{90^\circ} g(\theta) d\theta = 1.91, \]  
(21)

where again Eq. (6) has been assumed for the diffusion distribution. The remaining small discrepancy is due to the imperfect nature of the fit of Eq. (6) to the angular distribution data. Henceforth it will be assumed that: \( d_s = 2.00.\)

C. Thomson-Whiddington law and the mean energy

The square of the mean energy of a beam emerging from a target layer will approach zero asymptotically as the thickness of the slab approaches infinity. The use of the Thomson-Whiddington law to describe this quantity is based upon fitting a straight line to the initial, approximately linear, part of the curve. The data points in Fig. 6(a) show the Monte Carlo results for \( \langle u \rangle_F \) at five different incident energies. The solid curves were generated using the fit

\[ \langle u \rangle_F^2 = b \cdot (\rho x)^2 - c \cdot (\rho x) + 1.00, \]  
(22)

where \( b = 5999 \cdot T_0^{-3.148} \text{[mg cm}^{-2}\text{ cm}^4\text{]} \) and \( c = 130.1 \cdot T_0^{-1.559} \text{[mg cm}^{-2}\text{ cm}^2\text{]} \). The two-dimensional correlation coefficient for the fit was \( R^2 = 0.997.\) The broken curves in the figure correspond to linear fits (fitting for \( \langle u \rangle_F^2 > 0.4).\) The corresponding Thomson-Whiddington constants \( (C) \) are presented in Table I, along with the values presented by Birch and Marshall \( (C_{BM}).\) The linear relation is good in the region of fitting. There is, however, a failure to describe the onset of the asymptotic regime apparent in the data \( \langle \langle u \rangle_F^2 < 0.4 \). The Thomson-Whiddington range, as defined by \( C, \) is well described \( (R^2 = 0.999) \) by

\[ R_{TW} = 0.0119 \cdot T_0^{1.513} \text{[mg cm}^{-2}\text{].} \]  
(23)

The exponent of \( T_0 \) is very close to the energy dependence of the extrapolated ranges of Katz and Penfold. \( \text{Their predictions were } 1.446 \text{ at } T_0 = 150 \text{ keV and } 1.551 \text{ at } T_0 = 50 \text{ keV.}\) The fall of \( \langle \langle u \rangle_F^2 \) with penetration depth for aluminium, molybdenum, rhenium and tungsten are presented in Fig. 6(b). The rate of fall is only weakly dependent on the atomic number of the target material. The fall of \( \langle u \rangle_M^2 \) for aluminium, molybdenum, rhenium and tungsten are presented in Fig. 7(b). There is a more sizable variation between materials, with aluminium showing the largest discrepancy with respect to the tungsten results. The fall of \( \langle u \rangle_M^2 \) with depth in tungsten, in the range \( 50 < T_0 < 150 \text{ keV, } \) displayed in Fig. 7(a), is well described \( (R^2 = 0.996 \text{ in two dimensions}) \) by

\[ \langle u \rangle_M^2 = -a \cdot (\rho x)^3 + b \cdot (\rho x)^2 - c \cdot (\rho x) + 0.61, \]  
(24)

where \( a = 15859 \cdot T_0^{-3.903} \text{[mg}^{-3}\text{ cm}^6\text{]}, \) \( b = 1016 \cdot T_0^{-2.544} \text{[mg}^{-2}\text{ cm}^4\text{]} \) and \( c = 32.5 \cdot T_0^{-1.249} \text{[mg}^{-1}\text{ cm}^2\text{].}\)

D. First- and multiple-pass frequencies

The first- and multiple-pass frequencies for 100 keV incident electrons are shown in Figs. 8(a) and 8(b) for tungsten,

![Diagram](image_url)

**Fig. 5.** Angular distribution, \( g(\theta) \), for a 100 keV incident electron energy, at a density-scaled depth of 5.79 mg cm\(^{-2}\) (Monte Carlo data). The predictions of Eq. (6) also shown (solid curve).

**Fig. 6.** \( \langle u \rangle_F^2 \) as a function of penetration for (a) five initial electron energies and for (b) aluminium, molybdenum, rhenium and tungsten at a 100 keV incident energy (Monte Carlo data). The solid curves are polynomial fits using Eq. (22) and the dotted lines are the Thomson-Whiddington predictions, the values of \( C \) appearing in Table I.
rhenium, molybdenum and aluminium. The tungsten (Z = 74 and ρ = 19.3 mg cm⁻³) and rhenium (Z = 75 and ρ = 21.0 mg cm⁻³) results are essentially identical, as expected for two elements differing by only one unit in atomic number and having very similar densities. Molybdenum (Z = 42 and ρ = 10.3 mg cm⁻³) shows a marked departure from tungsten. Aluminium (Z = 13 and ρ = 2.7 mg cm⁻³) shows a considerable difference. The shape of the curve is sensitive to Z. The plural scattering, multiple-scattering and diffusion regions have been identified with an inflection, a linear portion and an exponential fall in ηₘ, respectively. The extent of the plural and multiple-scattering regions is expected to diminish with increasing Z and this is confirmed by the graphs. Tungsten is dominated by an exponential-looking rate of fall, molybdenum has a fairly large linear portion and aluminium exhibits a definite initial inflection. The lack of a clear linear portion in high-Z targets means that an extrapolated range (Rₑₑ) cannot be unambiguously defined. For aluminium, a low-Z target, an extrapolated range can be found unambiguously. The solid line on Fig. 8(a) shows a fit to the linear portion: the extrapolated range was 12.7 mg cm⁻². Katz and Penfold presented Carlvik’s experimental value of 13.0 ± 0.5 mg cm⁻² for this range, consistent with the value derived here. Further, using a continuous-slowing-down approximation (CSDA) range of 18.7 mg cm⁻² for 100 keV electrons incident on aluminium, the method of Tabata, Pedro, and Shinoda predicts an extrapolated range of 12.4 mg cm⁻², also close to the value found in this work.

The first- and multiple-pass frequencies are presented in Figs. 9(a) and 9(b) for tungsten and electron energies of 50, 80, 100, 120 and 150 keV. The solid curves are the best fits making use of Eqs. (11) and (14)–(16). The data are suitably described by the forms of these equations with the parameter Γ held constant, independent of T₀. The values of the constants found, for tungsten, were Γ = 1.753, K = 18.0 and Bₐ = 0.584. The value of the surface backscatter, B₀, was calculated from B₀ = ηₘ(x=0) and found to take the value 0.50.

The values of parameters quoted were determined from the data for the first-pass (η₊) and multiple-pass frequencies (ηₘ = η₊ + η₋). The solid squares in Fig. 10 depict η₊ and the empty squares η₋ for 100 keV electrons incident on tungsten. The curves through these points are the fit using Eq. (12) and Eq. (13) and the parameter values quoted above. The fit is good, despite these parameters having been determined from the sum of η₊ and η₋, not their independent values. This therefore validates the use of the geometric series approach. Both η₊ and η₋ are modeled correctly.

![Fig. 7](image-url)  
**Fig. 7.** ⟨μ⟩ₘ as a function of penetration for (a) five initial electron energies and for (b) aluminium, molybdenum, rhenium and tungsten at a 100 keV incident energy (Monte Carlo data). The solid curves are polynomial fits using Eq. (24).
E. Energy distribution

The shape of the CPF functions of 100 keV incident electrons at density-scaled depths of 1.93, 5.79 and 11.58 mg cm$^{-2}$ are presented in Fig. 11. The data points (circles) correspond to the Monte Carlo results binned in 2 keV increments. The solid curves are linear interpolations in $u$ and $x$. Unsurprisingly, the multiple-pass CPFs are broader than their first-pass counterparts and exhibit lower mean values of $u$. Note, however, that the first-pass component becomes rapidly more broad as it penetrates. Also, note the long low energy tail, present partly due to the generation of $\delta$ rays, the so-called Landau tail, but also arising from the distribution of path lengths of electrons at a given depth. The maximum cutoff in $u$ is absolute (rather than asymptotic) and is present due to the CSDA component of energy loss in EGSnrc. This cutoff decreases linearly with depth.

The CPF functions at a density-scaled depth of 3.86 mg cm$^{-2}$ are presented in Fig. 12 for five incident electron energies. Not only are less energetic electrons less penetrative, they also, as these results demonstrate, exhibit a broader distribution in energies at the same depth. This is despite possessing an essential identical angular distribution. The broken curves are linear interpolations of the Monte Carlo data at the relevant incident energy. The solid curves are the predictions using Eq. (17) and Eq. (18) with Monte Carlo data for $T_0=100$ keV, and linear interpolation in $x$ and $u$. The scaling law for extrapolation to arbitrary energy performs well.

Example data (circles) of the CPFs of rhenium, molybdenum and aluminium are presented in Fig. 13, at similar values of density-scaled depth. The solid curves are the predictions using linear interpolation in $u$ and $x$, between tungsten data. The CPF of rhenium is seen to very closely match the interpolation prediction in tungsten. Molybdenum shows a discernable discrepancy with the prediction based on tungsten and in aluminium the agreement is poor. This is unsurprising, as the ratio of elastic to inelastic scatter varies markedly as $Z$ decreases and the different balance of scattering mechanism inevitably affects the character of energy loss with depth. The shift of the CPF curve for aluminium towards lower energies with respect to the tungsten curve is consistent with Cosslett and Thomas’ experimental results for aluminium and gold, with incident electrons of 18 keV.

IV. DISCUSSION

The results of Sec. III A demonstrate that diffusion is reached rapidly, rapidly enough to assume, to a reasonable
approximation, that it is instantaneous. The scattering angle of an electron was also shown to be sufficiently independent of its kinetic energy to factorize the dependence on the \( u \) dependence. These facts allow the calculation of a simple multiplicative path-length correction for high-\( Z \) materials, using Eq. (7), which is (approximately) valid for all incident energies of interest \((50 < T_0 < 150 \text{ keV})\). This diversion correction, \( d_u \), was found in Sec. III B to take an approximately constant value of 2.

The analysis presented in Sec. III C showed that the Thomson-Whiddington law provided a good approximation for \( \langle u(x) \rangle_F \) only when \( \langle u \rangle_F^2 > 0.4 \). Nevertheless, since the first-pass survival frequency \( \eta_F \) declines more rapidly with depth than \( \langle u \rangle_F \) (Sec. III D), the Thomson-Whiddington law provides a fair approximation at the depths where \( \eta_F \) is appreciably large. The values of the Thomson-Whiddington constants presented in Table I correspond fairly closely with the value quoted by Birch and Marshall, which are used in several x-ray spectra models.\(^7\)\(^-\)\(^10\) It is unsurprising that their values are somewhat different to those of this work, as they were derived from the extrapolated range data summarized by Katz and Penfold\(^1\) based on low-\( Z \) materials, predominantly aluminium. Figure 6 showed that there is some deviation of the aluminium data from the results of tungsten. In addition, while it is perhaps legitimate to identify the extrapolated range with the Thomson-Whiddington range for low-\( Z \) materials, care should be taken in doing so for high-\( Z \) materials like tungsten, for which, as Fig. 8 clearly shows, there is no discernable linear portion.

There remain several ways to model the joint frequency density and hence the penetration of electrons into a target. Four of these are, in increasing order of sophistication,
transport and certain parameters derived in this work, such as $R_{ex}$ for aluminum, have agreed with experimental results. It seems likely, in any case, that the beam transport results presented here are a great quantitative improvement on the methods hitherto used in the models of x-ray spectra where the Thomson-Whiddington relation has been used.

The results presented here for the survival frequencies and CPFs should be applicable for heavy elements of atomic numbers similar to tungsten. They may also be of some applicability for elements of moderate Z such as molybdenum. It should be noted that there are x-ray tubes in use within healthcare that possess anodes made of elements that are not high-Z materials. The majority of such tubes are used in mammography. The tube potentials used in these cases are less than 50 kVp, outside of the range investigated here. If the approach adopted here were to be repeated and applied to mammographic energies and lower Z materials, careful consideration would have to be taken of the consequences of the less rapid onset of diffusion.

V. CONCLUSION

The concept of the joint frequency density, describing electron penetration in a target, was introduced and decomposed into survival frequencies and conditional probability functions. These components were calculated using Monte Carlo techniques for electron beams of energies of 50, 80, 100, 120 and 150 keV penetrating into a thick tungsten target. Results for low and medium atomic number targets were compared for 100 keV beams. A combination of parameterization in the form of equations, interpolation in energy fraction and depth, and extrapolation in incident beam energy, were used to generalize the results. The results of this paper allow the more accurate treatment of the penetration of electrons into high-Z targets, for use in models of x-ray spectra generation.

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