I. INTRODUCTION

Knowledge of the energy spectrum produced by an x-ray tube is crucial, as it affects patient dose and imaging performance. There is a long history of attempts to model such spectra. As far back as 1923, Kramers derived a formula for the energy spectrum using the correspondence principle and classical electrodynamics. This has had some success at describing empirical results. A rigorous quantum mechanical calculation by Bethe and Heitler (1934) soon followed. Both Kramers’ and Bethe and Heitler’s results were only immediately applicable to thin targets. The anode in a medical x-ray tube is thick compared to the penetration depth of incident electrons. Unsworth and Greening (1970) had some success with developing a thick target model based on Kramers work, and Soole calculated a target attenuation correction later in that decade, showing a more empirical approach to the treatment of the bremsstrahlung cross section. In 1978, Birch and Marshall helped towards making spectral calculation a common tool in medical physics with an approach which has come to be known as semiempirical. In the face of discrepancies between theoretical calculations and observation, following the tack of Soole, they abandoned derived formulas for the bremsstrahlung cross section and modeled this using an empirical function. They could make fairly accurate predictions for a variety of target angles, tube potentials and filtrations. A decade later, Iles (1987) developed a model that included backscatter and that author also obtained reasonable agreement with x-ray spectra. Tucker, Barnes and Chakraborty (1991) employed essentially the same approach as Birch and Marshall, their derivation following the notation of Evans. They fitted their model to a range of spectra data with a new parameterization and treated the characteristic spectrum in a more complicated manner. This model has been widely used.

There are several computer codes available for calculating x-ray spectra. The Institute of Physics and Engineering in Medicine (IPEM) has produced a user-friendly spectrum calculator available on a CD ROM (IPEM Report 78), based on the Birch and Marshall model. A purely empirical algorithm, TASMP, also exists and has been implemented in a freely available code, designated SPEKTR, in the MATLAB environment. There are many papers comparing various models, to each other and to experimental data: for example Meyer et al., Ay et al., or Bissonnette and Schreiner. Such a comparison is not the primary aim of this paper. Here, this author returns to the physical principles underlying these
models, something that to this author’s knowledge, with the exception of Blough et al.\textsuperscript{20} in the field of mammography, has been neglected in the last two decades.

The approaches referenced above, excepting TASZIP, make use of the Thomson-Whiddington relation to describe the penetration of electrons into a thick anode. This is a crude approximation that ignores backscatter and straggling. With increasingly accurate low-energy electron transport in Monte Carlo algorithms, there is the opportunity to abandon the crudity of the Thomson-Whiddington relation. While a full Monte Carlo treatment of the problem is possible, and in the near future computing power and carefully optimized variance reduction may allow it to become routine,\textsuperscript{21} there is much to be gained, in insight and calculation burden, from splitting the problem into two parts: electron penetration and x-ray emission. The first of these was treated in Part I of this work using Monte Carlo methods.\textsuperscript{22} Here, an analytic approach is applied to the second, using theoretical bremsstrahlung cross sections and the results of Part I. The characteristic contribution to the x-ray spectrum is treated here in an intuitive, but more approximate manner. This was considered appropriate to the much smaller contribution to the tube output it provides compared to bremsstrahlung. This would not be true in mammography applications, but these use tube potentials below those explored here.

II. THEORY

A. Formalism for thick-target bremsstrahlung

Heitler defined a bremsstrahlung cross section, \( \sigma \), for the emission of a photon from an electron of kinetic energy \( T_i \), in terms of a differential cross section with the units of area: \textsuperscript{23}

\[
\sigma = \int \Phi(hv,T_i) \frac{d(hv)}{T_i},
\]

where \( \Phi(hv,T_i) \) is the differential cross section for emission in the energy \( hv \) to \( hv + d(hv) \). The number of bremsstrahlung photons of energy \( hv \) to \( hv + d(hv) \) emitted in a target is then

\[
N_{\text{emit}}(hv) \cdot d(hv) = \left( d_n \int_0^\infty dx \int_{hv/T_i}^1 \frac{\Phi(hv,T_i(u))}{T_i(u)} f_{\Lambda,T_0}(u,x)du \right) \cdot d(hv),
\]

where \( N_{\text{emit}}(hv) \) is the number density of photons emitted and \( f_{\Lambda,T_0}(u,x) \) is the “joint frequency density.” This joint frequency density is the number density of electrons that occupy a position at depth \( x \), with a fraction \( u \) of their incident kinetic energy, \( T_0 \). This was calculated in Part I of this work.\textsuperscript{22} The constants \( n \) and \( d_n \) are the number density of target atoms and the mean diversion, respectively, where the diversion, calculated in Part I to take a value 2.0, is the ratio of an electron’s path length to its penetration depth. The superscripts on the joint frequency, \( \Lambda \) and \( T_0 \), refer to the lower and upper cutoffs in electron energy, respectively. These superscripts will be dropped subsequently to simplify notation, but should be taken to be implicitly implied.

For a target that is not thin, the spectrum emerging from the target will not correspond to that emitted within it, due to filtration by the target material. The filtered spectrum may thus be written in terms of a filtered number density

\[
N_{\text{filt}}^{\text{em}}(hv,\theta) = d_n \int_0^\infty dx \int_{hv/T_i}^1 \frac{\Phi(hv,T_i(u))}{T_i(u)} f(u,x)F(hv,x,\theta)du,
\]

where \( F(hv,x,\theta) \) is the fraction of photons of energy \( hv \) that escape the target, given that they are emitted at a depth \( x \), at a takeoff angle \( \theta \) with respect to the target. In Part I of this work, it was established that an electron penetrating into a thick target may be treated as if it suffered no lateral deflection, from the perspective of the origin of x-ray production. This scheme is illustrated in Fig. 1, where the path length of a bremsstrahlung photon in the target is calculated using the depth at which it is emitted, \( x \), and simple geometry. This allows the escape fraction to be approximated as

\[
F(hv,x,\theta) = \exp(-\mu_x(hv)x \csc \theta),
\]

where \( \mu_x(hv) \) is the attenuation coefficient for a photon of energy \( hv \) in tungsten.

The observed photon number density, however, will differ from this filtered density due to added filtration and the geometry of detection. Below the MeV energy range bremsstrahlung emission is fairly isotropic and, in a thick high-Z target, much of the residual anisotropy is blurred out due to multiple elastic scattering. The observed number density of bremsstrahlung may therefore be approximated by

\[
N_{\text{obs}}^{\text{em}}(hv,\theta) = GH(hv) \left( d_n \int_0^\infty dx \int_{hv/T_i}^1 \frac{\Phi(hv,T_i(u))}{T_i(u)} \right) \times f(u,x)F(hv,x,\theta)du,
\]

where

\[
G = \frac{A_d}{4\pi d^2} \quad \text{and} \quad H(hv) = \exp\left(-\sum_i \mu_i(hv)t_i\right)
\]

with \( \mu_i(hv) \) the linear attenuation coefficient for a material, \( i \), of thickness, \( t_i \), \( d \) is the focus-to-detector distance and where \( A_d \) is the detection area. The assumption has been made that photons scattered in the target and filtration do not contribute
to the measured spectrum. This is not unreasonable since Compton scattering shows a broad angular distribution at relevant tube potentials and the Rayleigh scattering cross section is relatively small.

To calculate photon spectra, expressions for the bremsstrahlung cross section are needed. These have been derived in scientific literature, in differing approximations.

### B. Kramers bremsstrahlung cross section

The Kuhnlenkampff-Kramers relation, describing the bremsstrahlung number density spectrum, may be expressed as

$$\lambda_{KK}(h\nu) = C_{KK}Z \left(1 - \frac{h\nu}{T_0}\right),$$  

(6)

where $C_{KK}$ is a constant and $Z$ is atomic number. The Kramers-Kuhnlenkampff relation was shown to have a degree of experimental validity by Kuhnlenkampff and co-workers and is derived from Kramers’ classical theory, under a number of assumptions, including nonrelativistic electrons and the Thomson-Whiddington law. Kramers, in fact, derived a more general classical result for the bremsstrahlung cross section itself. Kramers’ general result, in terms of the formalism developed here, may be expressed as

$$\Phi^{Kram}(h\nu, T_i) = \frac{16\pi}{3} \Phi \frac{\alpha^2 \lambda_0^2 T_i}{\beta_i^3 r_0^2 h\nu},$$

(7)

where $\lambda_0$ is the Compton wavelength, $\alpha = h/mc^2$, $\alpha$ is the fine-structure constant and $\Phi = Z^2 r_0^2 \alpha$, with $r_0$ being the classical electron radius. This will be referred to as the Kramers model, but should not be confused with the predictions of Eq. (6). The region for validity of this result is the classical limit where the scattering is strong: that is for a high-$Z$ material and low electron velocities. This may be expressed quantitatively in the conditions

$$2\pi\alpha Z / \beta_i = 2\pi \xi_i \gg 1 \quad \text{and} \quad 2\pi\alpha Z / \beta_f = 2\pi \xi_f \gg 1,$$  

(8)

where $\beta_i$ and $\beta_f$ are the velocities of the electron before and after emission, respectively, in units of the speed of light.

### C. Bethe-Heitler bremsstrahlung cross section

The theoretical result for the bremsstrahlung cross section in the strong scattering limit has also been established. Bethe and Heitler calculated nuclear-bremsstrahlung using early relativistic quantum theory and the Born approximation. Consider an electron of initial energy, $E_i$, corresponding momentum, $p_i$, kinetic energy, and rest-mass energy $mc^2$. Following the emission of a bremsstrahlung photon, this electron will have a final energy, $E_f = E_i - h\nu$, a corresponding momentum, $p_f$, and a kinetic energy, $T_f$. The Bethe-Heitler (BH) result for the bremsstrahlung cross section may then be expressed as

$$\Phi^{BH}(h\nu, T_i) = \Phi \frac{T_i}{h\nu} \frac{1}{p_i^{2c}} \left[4E_iE_f - 2E_i\frac{p_f^2c^2 + p_i^2c^2}{p^2c^4}T_i + \right.$$

$$\left. + m^2c^4 \left(\frac{e_iE_i}{p_i^3c^3} + \frac{e_fE_f}{p_f^3c^3} + \frac{e_iE_i}{p_epc}\right) + \right.$$

$$\left. + \frac{m^2c^4}{2pcp_e} \left(\frac{E_i^2}{p_i^3c^3} + \frac{E_f^2}{p_f^3c^3}\right) (E_iE_f + p_i^2c^2p_f^2c^2) \right]$$

$$+ \left(\frac{8E_iE_f}{3p_cp_e} + \frac{h\nu^2}{3p_i^3c^3} \left(E_i^2E_f^2 + p_i^2c^2p_f^2c^2\right) \right)$$

$$- \left(\frac{E_iE_f}{p_i^3c^3} + \frac{2h\nu E_i}{p_f^3c^3} \right),$$

(9)

where

$$L = 2 \ln \left(\frac{E_i + p_f c}{mc^2h\nu}\right),$$

$$e_i = 2 \ln \left(\frac{E_i + p_i c}{mc^2}\right),$$

and

$$e_f = 2 \ln \left(\frac{E_f + p_f c}{mc^2}\right).$$

This is a rather cumbersome result. However, in the nonrelativistic limit, the above expression simplifies drastically, to give

$$\Phi^{nrBH}(h\nu, T_i) = \lim_{\beta_i \to 0} \Phi^{BH}(h\nu, T_i)$$

$$= \frac{16mc^3}{3h\nu} \ln \left(\frac{T_i}{h\nu} + \sqrt{\frac{T_i}{h\nu} - 1}\right).$$

(10)

This nonrelativistic Bethe-Heitler (nrBH) expression has been used at tube potentials appropriate to mammography, of less than 50 kVp. Unfortunately, at higher tube potentials, relevant to medical applications outside of mammography, the approximation becomes inaccurate. Yet, over this entire range, the kinetic energy of an electron striking an x-ray target remains smaller than its rest-mass energy. In the region of interest, $T_i < mc^2$, the full Bethe-Heitler result remains well approximated by a simpler formula, derived in Appendix A. The semirelativistic (srBH) result is

$$\Phi^{srBH}(h\nu, T_i) = \frac{2}{3h\nu} \frac{T_i}{p_i^2c} \left[4E_iE_f - 7p_ePC_c\right],$$

(11)

where $L$ takes the relativistic form delineated above. Figure 2 compares the full BH, nrBH and srBH cross sections for an electron kinetic energy, $T_i = 150$ keV.

### D. Coulomb correction to Bethe-Heitler

The Bethe-Heitler result was obtained at the “Born level.” This approximation assumes that the wave functions of the electron scattering off a nucleus are the plane waves for the electron at infinity before and after the collision. The approximation is applicable where the initial and final electron
is highly energetic and where the perturbing potential is weak. This condition may be expressed in the conditions

\[ 2\pi \xi_i \ll 1 \quad \text{and} \quad 2\pi \xi_f \ll 1, \quad (12) \]

with \( \xi_i \) and \( \xi_f \) having the same definitions as discussed above in relation to Kramers’ calculation. For tungsten \((Z=74)\) and a 100 keV electron \((\beta_i=0.5)\),

\[ 2\pi \xi_i \sim 6. \]

The Born approximation is not then valid for tungsten and the tube potentials of interest. The long-range Coulomb field of a high-Z nucleus distorts the wave function of the electron from its initial plane wave on its approach. However, the interaction is not sufficiently strong to be confident that Kramers’ formula is accurate. Proposals have been made to approximately correct the Born approximation using a multiplicative term called the Sommerfeld factor. Sauter proposed multiplying the Born cross section by a simple factor

\[ f_S(\xi_i, \xi_f) = \frac{2\pi \xi_i}{e^{2\pi \xi_i} - 1} \cdot \frac{2\pi \xi_f}{1 - e^{-2\pi \xi_f}}, \quad (13) \]

such that

\[ \Phi_{SBH}(hv/T_i) = f_S(\xi_i, \xi_f) \Phi_{BH}(hv/T_i), \quad (14) \]

where \( \Phi_{SBH} \) is the Sauter-corrected cross section. The argument is that this renormalization approximately corrects for the focusing effect of the Coulomb field. This recipe appeared as the suggested correction in the second edition of Heitler’s book on quantum radiation. Blough et al. used this Sommerfeld factor to calculate mammographic x-ray spectra, referencing the second edition of Heitler. By the third edition of that book, however, appearing in 1954, the Sommerfeld factor had been discarded in favor of an alternate factor, first proposed by Elwert. This has had some success in correcting the Born level results elsewhere and will be used in this work. The Elwert factor is defined

\[ f_E(\xi_i, \xi_f) = \frac{\xi_f}{\xi_i} \frac{1 - e^{-2\pi \xi_i}}{1 - e^{-2\pi \xi_f/\beta_i}} \cdot \beta_f. \quad (15) \]

A simple theoretical argument for the use of this factor is attempted in Gould. The Sauter factor possesses the same dependence on \( h\nu/T_i \) as the Elwert factor, but has a different, \( T_i \)-dependent, normalization. The Sommerfeld factors affect the emission cross section such that it is no longer nonzero at \( h\nu = T_i \), hardening the emission spectrum. More recently, Avdonina and Pratt have demonstrated that a better agreement with the full partial wave calculation is obtained if Elwert factor is reinterpreted. They suggest the replacement \( \beta_{if}\rightarrow p_{if}/mc \) in this factor.

Figure 3 compares the Bethe-Heitler (BH), Sauter-corrected Bethe-Heitler (SBH), Elwert-corrected Bethe-Heitler (EBH), and modified Elwert-corrected Bethe-Heitler (MEBH) and Kramers bremsstrahlung cross sections, against the fraction of electron kinetic energy emitted \((T_i=100 \text{ keV and } Z=74)\). NIST cross section also shown for comparison.
\[ r \sim \frac{\hbar c}{q}, \]

where \( r \) is a length scale, \( \hbar \) is Planck’s constant, \( c \) is the speed of light and \( q \) is the momentum gained by the atom through the scatter. This momentum takes a minimum value \( q_{\text{min}} = p_i - p_f - k \),

where \( k \) is the momentum of the bremsstrahlung photon. For an electron with a kinetic energy of 100 keV scattering from a tungsten atom, \( r \) approaches \( a \) when \( hv/T_i \) falls below \( \sim 0.2 \). For emission energy fractions much above this, the scattering electron is probing deep within the target atom and the effective charge of the target nucleus is well approximated by the nuclear charge. The MEBH result provides a reasonable approximation here. For softer emissions, the outer electrons partially screen the nucleus reducing the bremsstrahlung cross section appreciably. In a thick-target x-ray tube, a negligible proportion of photons with energies below 10 keV escape from the tube. For incident electron kinetic energies, \( T_0 \), in the range 50–150 keV, most of the photons observed will possess energy fractions \( > 0.2 \) times the parent electrons’ kinetic energy, \( T_i < T_0 \). The relatively simple, semirelativistic MEBH formula may therefore provide a reasonable approximation to the NIST cross sections, for tube potentials and photon energies of interest here.

### E. Treatment of characteristic x-ray production

The calculation of the characteristic component of the x-ray spectrum is complicated by the fact that these x rays are generated by two processes within the target: impact ionization and photoelectric absorption. The former, also called direct emission, occurs as the beam electrons penetrate into the target knocking K-shell electrons out of target atoms. The latter mechanism, the photoelectric absorption of photon energies above the K edge, will also result in indirect characteristic emission. Given the relatively small contribution of characteristic radiation compared with bremsstrahlung for electron energies of interest here, no rigorous attempt to model this contribution is attempted. Rather, using a simple calculation, the correct magnitude is predicted and an empirical adjustment is then applied.

The number density of K-fluorescent photons observed in the \( i \)th K-line transition, due to indirect emission, may be expressed as

\[ N_{\text{char}}(hv) = \delta(hv - h\nu_k) \int_{h\nu_k}^{T_0} d(h\nu') N_{\text{emit}}(h\nu') P_i, \]

where \( P_i \) is the probability an emitted photon results in an observed characteristic photon of the \( i \)th K line and \( h\nu_k \) is the K-edge energy. This probability may be decomposed into the various steps of the physical process, such that

\[ P_i = P_{\text{abs}} P_{\text{ion}} P_{\text{flu}} P_{\text{th}} P_{\text{obs}}. \]

The probability that an absorbed photon, of energy above the K edge, results in a K-shell ionization is approximately

\[ P_{\text{ion}} = f_k = \frac{r_k - 1}{r_k}, \]

where \( r_k \) is the K-edge discontinuity factor (\( \sim 4.4 \) for tungsten). The probability of a K fluorescence given a K-shell ionization (i.e., fluorescent yield) is

\[ P_{\text{flu}} = \omega_k = 1 - f_{\text{Auger}}, \]

where \( f_{\text{Auger}} \) is the fraction of ionizations that result in an Auger electron, the value of which is small for heavy elements. The probability of a K fluorescence in the \( i \)th line, given that a K fluorescence occurs, is in common with the notation written

\[ P_{\text{th}} = f_i, \]

the values of which are tabulated in experimental literature. The probability for detection of a K fluorescence, ignoring the attenuation of the fluorescence in the target, is simply

\[ P_{\text{obs}} = GH(hv), \]

where \( H(hv) \) and \( G \) take the forms discussed previously.

The direct component of characteristic radiation remains to be described. This component of fluorescence is generated close to the target surface. It has also been shown experimentally that the ratio of direct to indirect characteristic production does not vary greatly, for a given material, over a range of tube potentials. Therefore,

\[ N_{\text{char}}(hv) \approx GH(hv)(1 + P)(1/2f_i \omega_k) \times \delta(hv - h\nu_k) \int_{h\nu_k}^{T_0} N_{\text{emit}}(h\nu)d(h\nu), \]

with only one free parameter, \( P \), appearing, the remainder being prescribed either by experiment or theory. This formula neglects self-attenuation within the target. This should be a good approximation for the direct component. This is expected to not be such a good assumption for the indirect component, as the mean production depth is deeper, due to the penetrative power of photons of energy above the K edge. Although the attenuation would be expected to be non-negligible, it is likely to not be a very large effect. This attenuation correction will be approximately accounted for by absorption into a lower “effective” value for \( P \). It should be noted, however, that the large attenuation of the indirect component at very small takeoff angles will not be described correctly. Semiempirical approaches, such as that of Tucker et al., also fail to accurately describe the observed trend at small takeoff angles.

### F. Joint frequency density

The joint frequency density describes the number density of electrons at a depth \( x \), with a fraction of their initial kinetic energy, \( u \). The joint frequency density may be decomposed...
into a planar survival frequency, $\eta_p(x)$, and a conditional probability function (CPF) describing the energy distribution, $P(u|x)$. Doing so,

$$ f(u,x) = \eta_p(x)P(u|x). $$

(24)

$P(u|x)$ is the probability density of an electron possessing a kinetic energy $T_f = uT_0$, given that it is at a depth $x$ within the target. This CPF is normalized such that

$$ \int_\Lambda^1 P(u|x)du = 1, $$

(25)

where $\Lambda$ is a cutoff in energy fraction, set to 10 keV/$T_0$ in this work. As described in Part I, this expression may be further decomposed into first-pass ($F$) and multiple-pass components ($M$), such that

$$ f(u,x) = \eta_F(x)P_F(u|x) + \eta_M(x)P_M(u|x). $$

(26)

In Part I, four approximations were proposed. These were, in order of increasing sophistication

$$ f_1(u,x) = \delta[u - u_{TW}(x)], $$

(27a)

$$ f_2(u,x) = \eta_F(x)\delta[u - u_{TW}(x)], $$

(27b)

$$ f_3(u,x) = \eta_F(x)\delta[u - \langle u(x) \rangle_F] + \eta_M(x)\delta[u - \langle u(x) \rangle_M], $$

(27c)

$$ f_4(u,x) = \eta_F(x)P_F(u|x) + \eta_M(x)P_M(u|x). $$

(27d)

Empirical expressions for $u_{TW}(x)$, $\langle u(x) \rangle_F$ and $\langle u(x) \rangle_M$ were presented in Part I.22

III. METHOD

A. Calculation of spectra

For explicitness, the expressions for the various number densities of observed photons, per incident electron, will be quoted in detail. They are amalgamations of results of Eqs. (3), (4), (7), (11), (15), and (23) and have the forms,

$$ N_{\text{Kram}}^{\text{obs}}(hv) = N_FGH(hv)d_n \times \int_0^\infty dx f(hv,x,\theta) \int_0^1 du \left( \Phi \frac{2}{3} \frac{1}{hv} \frac{1}{p_i^2 c^2} \right) \left[ 4E_iE_fL - 7p_i c p_f c \right] f(u,x), $$

(28a)

or

$$ N_{\text{Kram}}^{\text{obs}}(hv) = N_FGH(hv)d_n \times \int_0^\infty dx f(hv,x,\theta) \int_0^1 du \left( \Phi \frac{2}{3} \frac{1}{hv} \frac{1}{p_i^2 c^2} \right) \left[ 4E_iE_fL - 7p_i c p_f c \right] F^\text{obs}_{\text{char}}(hv'). $$

(28b)

TABLE I. Energies of K edge in tungsten and major associated characteristic lines (see Ref. 8).

<table>
<thead>
<tr>
<th>K edge</th>
<th>Energy, $\alpha_1$</th>
<th>Energy, $\alpha_2$</th>
<th>Energy, $\beta_{1,3,5}$</th>
<th>Energy, $\beta_{2,4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>keV</td>
<td>keV</td>
<td>keV</td>
<td>keV</td>
<td>keV</td>
</tr>
<tr>
<td>-------</td>
<td>-------------------</td>
<td>-------------------</td>
<td>----------------------</td>
<td>----------------------</td>
</tr>
<tr>
<td>69.5</td>
<td>59.3</td>
<td>58.0</td>
<td>67.2</td>
<td>69.1</td>
</tr>
</tbody>
</table>

and

$$ N_{\text{srBH}}^{\text{obs}}(hv) = N_FGH(hv)d_n \times \int_0^\infty dx f(hv,x,\theta) \int_0^1 du \left( \Phi \frac{2}{3} \frac{1}{hv} \frac{1}{p_i^2 c^2} \right) \left[ 4E_iE_fL - 7p_i c p_f c \right] f(u,x). $$

(28c)

where $X$ is srBH, srMEBH or Kram and

$$ N_{\text{srBH}}^{\text{emit}}(hv) = N_Fd_n \times \int_0^\infty dx \int_0^1 du \left( \Phi \frac{2}{3} \frac{1}{hv} \frac{1}{p_i^2 c^2} \right) \left[ 4E_iE_fL - 7p_i c p_f c \right] f(u,x). $$

(29a)

and

$$ N_{\text{srBH}}^{\text{emit}}(hv) = N_Fd_n \times \int_0^\infty dx \int_0^1 du \left( \Phi \frac{2}{3} \frac{1}{hv} \frac{1}{p_i^2 c^2} \right) \left[ 4E_iE_fL - 7p_i c p_f c \right] F_{\text{srBH}}^{\text{emit}}(hv). $$

(29b)

The values of the experimental parameters used in this work, relating to characteristic radiation, are presented in Tables I and II.

TABLE II. Quantities used to describe characteristic x-ray spectrum in tungsten, values being taken from Ref. 8 except for $\omega_k$, taken from Ref. 35 and $r_k$, calculated from Ref. 36.

<table>
<thead>
<tr>
<th>Fraction, $\alpha_1$</th>
<th>Fraction, $\alpha_2$</th>
<th>Fraction, $\beta_{1,3,5}$</th>
<th>Fraction, $\beta_{2,4}$</th>
<th>$\omega_k$</th>
<th>$r_k$</th>
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<td>[keV]</td>
<td>[keV]</td>
<td>[keV]</td>
<td>[keV]</td>
<td></td>
</tr>
<tr>
<td>0.505</td>
<td>0.291</td>
<td>0.162</td>
<td>0.042</td>
<td>0.94</td>
<td>4.4</td>
</tr>
</tbody>
</table>
B. Implementation

The bremsstrahlung and characteristic spectra were generated by implementation of the above formulas in MATLAB (MathWorks Inc., Natwick, MA). In all cases, unless explicitly mentioned otherwise, values for the attenuation coefficients of the various materials were calculated by interpolation between data of the NIST XCOM database. The fits arrived at in Part I were used for the first and multiple-pass survival fractions. The first- and multiple-pass CPFs were obtained using Monte Carlo data sets, as described in that paper, where \( T_0 = 50, 80, 100, 120 \) or 150 keV. Where necessary for calculations, extrapolation was performed from the nearest value of \( T_0 \) using the method described in Part I.

The experimental data used in this work were from two x-ray tubes: a Toshiba T700 and a GE tube. With the first, Bhat et al. obtained and published central-axis photon spectra for 50, 80 and 100 kVp tube potentials. This tube possessed an anode angle of 12° and a 1.2 mm Al equivalent intrinsic filtration. The NIST XCOM attenuation coefficients were used for all materials in all cases except for when implementing the model of Tucker et al. and using IPEM Report 78. When implementing the model of Tucker et al., the intrinsic filtration of the Toshiba tube was assumed equivalent to the inherent filtration in their work. This was 2.38 mm of glass, 3.06 mm of oil and 2.66 mm of Lexan, stated as nominally equivalent to 1.2 mm of Al. When predicting spectra using IPEM Report 78, the inherent filtration was taken as Al and the attenuation coefficients within that software were used. The error on the spectral data assumed here corresponds to the 5% precision level that is mentioned in Bhat et al. In this work, however, these spectra were stripped of the characteristic component by linear interpolation of the continuous component. The published data were binned in 2 keV increments and here were re-normalized such that the area under the curve was one photon, after the stripping procedure, when integrated over the interval 15 to \( T_0 \) keV.

The data for the GE tube were taken from the Tucker et al. paper, in which those authors quote the experimentally derived data of Fewell, Shuping, and Hawkins for the bremsstrahlung and characteristic contributions to the output of such a tube. These outputs were quoted for tube potentials 70–140 kVp, on the central axis of the beam, at a 1 m focus-to-detector distance. The anode target angle was 10° and the inherent filtration consisted of glass, oil and Lexan, as described above. These three materials, along with the stated 1.5 mm added filtration of Al, were described using the parameterizations of attenuation coefficients presented in Tucker et al. when implementing their model. The glass, oil and Lexan were assumed replaceable by their nominal 1.2 mm Al in all other cases. Overall normalizations of the outputs of the model proposed here were accomplished by adjusting \( N_f \) (for the bremsstrahlung) and \( P \) (for the characteristic x rays).

![Fig. 4. MEBH spectral predictions for a 100 kVp tube potential (12°anode angle, W target and 1.2 mm Al filtration). Four different treatments of the joint frequency are shown (curves) and compared to the data of Bhat et al. (circles).](image)

IV. RESULTS

A. Treatment of joint frequency density

The four curves in Fig. 4 correspond to the predictions of an x-ray spectrum for the Toshiba tube operated at 100 kVp, with the four different treatments of the joint frequency. In each case, the (semirelativistic) MEBH bremsstrahlung cross section has been used. All but the fourth approach employ some form of \( \delta \)-function approximation in the CPF and therefore neglect energy straggling. The first treatment of the joint frequency, \( f_1(u|x) \), using Eq. (27a), neglects the survival fraction for an electron to penetrate to a designated depth, \( x \). This boosts the contribution from larger \( x \), corresponding on average to lower \( u \), thus softening the spectrum. Additionally, the softer multiple-pass component is discarded, hardening the spectrum. The net result, as can be gauged from the figure (dashed line), is a spectrum that is appreciably too hard. The second treatment, \( f_2(u|x) \), using Eq. (27b), includes the planar survival fraction, while neglecting the multiple-pass contribution. In this case, the spectrum is even harder (dotted line) and in poor agreement with the data. The third treatment, \( f_3(u|x) \), using Eq. (27c), included a multiple-pass component. Figure 4 shows that this results in an improved fit to the data (thin solid line). However, equating the energy fraction at depth to the mean value, \( \langle u(x) \rangle_M \), in the multiple-pass component, results in an underestimation of the high-energy contribution to the spectrum. The fourth approach, \( f_4(u|x) \), using Eq. (27d), the most complete treatment, shows the best agreement with the measured spectrum (thick solid line).

The prediction of the full treatment for a 100 kVp tube potential is shown again in Fig. 5 (solid line). The proportions of this bremsstrahlung number density that are due to first-pass (dashed line) and multiple-pass (dotted line) components are also shown. The former is harder than the latter. Both components are required to accurately describe the spectrum. In the remainder of this paper, a full treatment of the joint frequency is used, except in the semiempirical models of Tucker et al. and IPEM Report 78.
The mean emission depth spectrum is shown in Fig. 6 for a 100 kVp tube potential. In Part I of this work it was established that an electron beam of electrons with kinetic energies of 100 keV, incident on a tungsten target, reaches diffusion by a depth of \( \frac{1}{H_{11011}} \). Figure 6 demonstrates that for most of the x-ray spectrum, the mean depth of production exceeds this value. An overall path-length correction by the mean diversion at diffusion, \( d_x \), should therefore be approximately valid, as discussed in that paper. Also, as would be expected, both the soft and hard ends of the spectrum show lower mean emission depths than the middle range. The reason for the former is that lower energy photons are strongly filtered in the target. The reason for the latter is that few electrons penetrate to large depths with a high kinetic energy, due to inelastic scattering.

B. Treatment of bremsstrahlung cross section

The bremsstrahlung spectra data of Bhat et al. are shown in Figs. 7(a)–7(c), for tube potentials of 50, 80 and 100 kVp, respectively. Theoretical predictions, using the Kramers, BH and MEBH expressions for the bremsstrahlung cross sections, are also shown. The classical spectral predictions of Kramers are consistently too hard, except at the lowest tube potential. The Born level predictions of BH, by contrast, are consistently much too soft. The Elwert-corrected spectra are intermediate between these two extremes and show reasonable agreement with the data. The agreement is remarkably good, considering that no fitting process was involved, and the bremsstrahlung cross section is without free parameters. The agreement using interpolation of the NIST bremsstrahlung cross sections is also good, except for photon energies above the \( K \) edge, where the number of photons appears to be over-estimated.

The photon spectra of Bhat et al. are again shown in Figs. 8(a)–8(c), this time in comparison to the MEBH predictions and those of Tucker et al. and IPEM Report 78. All three models give a reasonable description of the data, although the Birch and Marshall model implemented in IPEM 78 con-
consistently results in slightly harder spectra. The spectra of the MEBH and Tucker et al. models are in close agreement. Table III presents the root mean square (rms) errors between the six bremsstrahlung models and the Bhat et al. data. Table IV presents the mean energies of those spectra. These figures confirm that the MEBH and NIST cross-section models developed here and that of Tucker et al. are in best agreement with the experiment.

C. Bremsstrahlung and characteristic x-ray outputs

The experimentally derived outputs of Fewell et al. for both the bremsstrahlung and the characteristic contributions to the absorbed dose are presented in Fig. 9. The Air Kerma predictions of the MEBH model (\(N_f=0.68\) and \(P=0.33\)), and those of Tucker et al. and IPEM Report 78, are also shown. The correspondence between the data and all the models is good, with discrepancies of less than a few percent for the model proposed here, over the whole potential range 70–140 kVp. There are several possible sources for the departure of \(N_f\) from unity. Uncertainties in the degree of filtration of the x-ray unit, and in the attenuation coefficients, may contribute. The neglect of anisotropy in the angular distribution of bremsstrahlung emission is another likely source. Path-length overestimation due to the assumption of instant diffusion probably plays some role, particularly for photons where \(hv/T_0\sim 1\), which, as is demonstrated in Fig. 6, are emitted closer to the surface. It is also not inconceivable that the NIST bremsstrahlung cross section genuinely overestimates that observed to some degree.

V. DISCUSSION

The prediction of the x-ray spectrum emitted from an x-ray tube is an important problem in medical physics. A number of models for the calculation of such spectra exist, are regularly used, and are likely to remain so. Despite being useful practical tools they have serious theoretical insufficiencies, a partial redress of which has been attempted in this work and its companion paper. The model developed here has proved remarkably successful. It is based on a more realistic treatment of electron penetration, using Monte Carlo derived results, and an abandonment of the semi-empirical treatment of the bremsstrahlung cross sections. The spectra at three tube potentials (50, 80 and 100 kVp) have been well described with the use of the full treatment of the joint frequency (see Sec. IV A) combined with the (semirelativistic) MEBH or NIST bremsstrahlung cross section (see Sec. IV B). Both the classical results of Kramers and the Born

---

**Table III.** The rms errors of six models of bremsstrahlung spectra, at three tube potentials, in comparison to the data of Bhat et al.

<table>
<thead>
<tr>
<th>Model</th>
<th>50 kVp</th>
<th>80 kVp</th>
<th>100 kVp</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kramers</td>
<td>1.6</td>
<td>1.6</td>
<td>1.5</td>
</tr>
<tr>
<td>BH</td>
<td>7.8</td>
<td>3.9</td>
<td>2.8</td>
</tr>
<tr>
<td>MEBH</td>
<td>1.5</td>
<td>1.0</td>
<td>0.7</td>
</tr>
<tr>
<td>NIST</td>
<td>1.3</td>
<td>1.1</td>
<td>0.9</td>
</tr>
<tr>
<td>Tucker et al.</td>
<td>1.4</td>
<td>0.8</td>
<td>0.6</td>
</tr>
<tr>
<td>IPEM 78</td>
<td>2.0</td>
<td>1.7</td>
<td>1.5</td>
</tr>
</tbody>
</table>

**Table IV.** Mean photon energies in six models of bremsstrahlung spectra, at three tube potentials. The corresponding mean energies obtained from the experimental data of Bhat et al. are also presented.

<table>
<thead>
<tr>
<th>Model or Data</th>
<th>Mean photon energy [keV]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>50 kVp</td>
</tr>
<tr>
<td>Kramers</td>
<td>30.8</td>
</tr>
<tr>
<td>BH</td>
<td>28.3</td>
</tr>
<tr>
<td>MEBH</td>
<td>30.2</td>
</tr>
<tr>
<td>NIST</td>
<td>30.6</td>
</tr>
<tr>
<td>Tucker et al.</td>
<td>30.0</td>
</tr>
<tr>
<td>IPEM 78</td>
<td>30.4</td>
</tr>
<tr>
<td>Bhat et al.</td>
<td>30.3</td>
</tr>
</tbody>
</table>
level quantum mechanical calculation of Bethe-Heitler were insufficient, even with the fullest treatment of electron penetration. The semiempirical predictions of Tucker et al. at the three examined tube potentials also fitted the spectral data well. The IPEM Report 78 predictions were consistently slightly harder, as has been observed elsewhere.\textsuperscript{15,19} The experimental comparisons suggest that the model developed in this work performs, essentially, as well as the model of Tucker et al. and, perhaps, slightly better than the Birch and Marshall model implemented in IPEM Report 78.

The output [\(\mu\text{Gy}/\text{mAs}\)] from x-ray tubes varies from one x-ray tube to another due to factors such as the electron beam focusing, anode material and angle, power supply characteristics and the inherent and added filtration. Even in a single tube, there will be variation in beam quality over time due to factors such as deterioration in the anode surface and anode material vaporization and deposition on the exit window. Yet, the correct variation of output with tube potential would be expected to be predicted by a realistic model and the magnitude of the output would be expected to be of the same order as that is observed. The model developed in this work correctly predicted the trends in output variation with tube potential for an example x-ray tube, and, with fairly natural values of the adjustable parameters \(N_f\) and \(P\), the absolute normalizations of both continuous and characteristic components (see Sec. IV C). The models of Tucker et al. and IPEM Report 78 perform similarly well.

The model proposed here is an advance in that the underlying physics is modeled more completely. The bremsstrahlung cross section of Tucker et al. contains four unphysical free parameters describing the shape of the distribution and a further two describing its normalization. The cross section of Birch and Marshall contains a total of five such unphysical parameters. The MEBH bremsstrahlung cross section contains no free parameters, except for the overall normalization factor introduced, which, in any case, is close to unity. A disadvantage with empirical or semiempirical treatments is that, because of the presence of unphysical parameters, confidence in the model may not be justified when it is applied to new or unconventional x-ray tube designs. The model proposed here may prove more successful in such situations.

Despite the success of this work, it is worth highlighting some of the assumptions and limitations of this model. X-ray emission has been assumed isotropic, and this has been assumed to be justified due to the relatively low electron energies and the high degree of elastic scatter in high-Z materials. The voltage ripple has been assumed to be negligible. Modern x-ray sets have an excellent constancy at the peak voltage so this assumption should be reasonable in most circumstances. The penetration depth of electrons in the anode has been assumed to be much smaller than the focal spot size.\textsuperscript{22} This may not be reasonable if this model is applied to low or moderate-Z materials, at high tube potentials and with large focal spot sizes. Furthermore, off-focus radiation has been ignored in this model. For a tungsten anode, approximately 50% of electrons incident on the target backscatter away from the target.\textsuperscript{22} In reality, some of these are likely to be re-focused back onto the target, the majority of those being re-incident away from the original focus. Here, however, these electrons are assumed lost. This should be a reasonable assumption as off-focal radiation may contribute less than 10% to the absorbed dose from a diagnostic x-ray beam.\textsuperscript{38} Furthermore, in this work the characteristic spectrum is treated simply, with no attenuation of the indirect component and an empirical adjustment to take impact ionization into account. This approach was considered sufficient since the characteristic component of an x-ray beam, except in mammography, contributes less than \(\sim 15\%\) of the absorbed dose.\textsuperscript{10,12,13} It should be noted, however, that the characteristic component will not be well-described at very small takeoff angles due to attenuation of the indirect component by the target material. Finally, the Monte Carlo calculations of electron penetration characteristics were based on a normally incident electron beam, and, consequently, the self-attenuation distance was set to \(x \csc \theta\). This assumption was also made in the model of Tucker et al., though not in that of Birch and Marshall. In the literature, the tilting of a tube, or a movement away from the central axis in the anode-cathode direction, has often been considered equivalent to an equal change in the anode angle.\textsuperscript{8,12,39} In most circumstances the electron beam has an oblique rather than normal incidence to the anode in an x-ray tube. This obliquity is usually small, but its consequences may need to be considered at large anode angles.

VI. CONCLUSION

A model for the calculation of the x-ray spectrum emerging from an x-ray tube was presented for tube potentials in
the range of 50–150 kVp. This model made use of theoretical results for the bremsstrahlung cross section instead of a semiempirical treatment and, consequently, involved fewer unphysical parameters in the bremsstrahlung cross section. A simple treatment of the characteristic spectrum was also proposed. The predicted spectra were in good agreement with data for investigated tube potentials (50–100 kVp) and trends in the continuous and characteristic components of tube outputs were correctly described (70–140 kVp). This work demonstrates that with a more thorough treatment of electron penetration and bremsstrahlung emission, semiempirical fixes are unnecessary and that there is a valid alternative to a full Monte Carlo treatment of the problem.

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APPENDIX A: THE SEMIRELATIVISTIC CROSS SECTION

In the energy region where the initial velocity of a scattering electron, \( \beta_e \), is not close to unity, the following approximations may be made:

\[
E_i \sim mc^2 + \frac{p_i^2}{2m}, \quad E_f \sim mc^2 + \frac{p_f^2}{2m},
\]

\[
e_i \sim \frac{2p_i}{mc} \quad \text{and} \quad e_f \sim \frac{2p_f}{mc}.
\]

The various contributions to the Bethe-Heitler bremsstrahlung cross section may then be approximated as

\[
Q_1 = 2E_i E_f \left( \frac{p_i^2 c^2 + p_f^2 c^2}{p_i c p_f c} \right) - m^2 c^4 \left( \frac{e_i E_i}{p_i c^3} + \frac{e_f E_f}{p_f c^3} \right)
\]

\[
= \left[ 2 + O \left( \frac{p_{	ext{eff}}}{m} \right) \right],
\]

\[
Q_2 = m^2 c^4 \left( \frac{e_i E_i}{p_i c p_f c} \right) \sim \left[ 4 + O \left( \frac{p_{	ext{eff}}}{m} \right) \right],
\]

and

\[
Q_3 = L \left[ \frac{8}{3} \frac{E_i E_f}{p_i c p_f c} + \frac{2}{3} \frac{E_i^2}{p_i c^3} (E_f^2 p_i^2 c^2 + p_f^2 c^2) \right]
\]

\[
+ \frac{m^2 c^4 h^2}{2p_i c p_f c} \left( E_i E_f + p_i^2 c^2 e_i - E_i E_f + p_f^2 c^2 e_f \right) + \frac{2h^2 E_i E_f}{p_i c p_f c^3} \left[ \frac{1}{3} - O \left( \frac{p_{	ext{eff}}}{m} \right) \right]
\]

Putting these contributions together, we arrive at the expression

\[
\Phi_{\text{BH}}(h, T_e) = \Phi_{\text{srBH}}(h, T_e) = \Phi \left[ \frac{T_i p_i}{3} \left( \frac{4}{3} - Q_1 - Q_2 + Q_3 \right) \right]
\]

The epithet “semirelativistic” (srBH) is applied since relativistic expressions for the kinematical variables shall be assumed in the remaining components.

6Electronic mail: gavin.poludniowski@icr.ac.uk
27E. Mainegra-Hing and I. Kawrakow, “Efficient x-ray tube simulations,”


